Averaged control of uncertain wave equations

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Abstract

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Résumé

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1. Introduction

We introduce and analyse the problem of recording the initial energy of a system of wave equations by observing the averaged value of the solutions on a suitable region. Depending on the form of initial data of the system, two types of results are obtained, requiring Geometrical control condition (GCC) to be satisfied either for a single equation or for all of them.

We refer to two types of obtained results as averaged and simultaneous observability, in accordance to the terminology in the control theory. The notion of averaged observability is weaker, as it assumes special kind of initial data, but also provides the result under weaker conditions. To our knowledge it has not been studied yet. Simultaneous observability has been considered already in [4] for a cascade system of wave equations, and in [1] for a system of equations coupled by zero order terms. The latter one analyse

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equations with same coefficients, while the former allows for the case of different velocities associated to each equation as well.

We obtain the proof of observability inequalities by tools of microlocal analysis, in particular by H-measures. These have been introduced independently by P. Gérard [7] and L. Tartar [8] (microlocal defect measures in the terminology of the former). We refer the reader to the mentioned articles for the properties of the H-measures (localisation, propagation, etc.) used in this note. Similar kind of arguments with the same tool have already been applied to study of control problems for hyperbolic equations [2,3] or systems [4].

In the next section we discuss the averaged observability problem for a finite system of wave equations distinguished by different coefficients, while simultaneous observability is the subject of section 3. We close the paper by relating the results obtained for the wave equation to a system of heat equations. We also point towards some open problems and future directions of research.

2. Averaged observability

We consider two problems for the wave equation determined by the same initial data and different coefficients entering them, with solutions denoted respectively by $u_1$ and $u_2$:

$$
\begin{align*}
\partial_{tt}u_i - \text{div} (c_i(t,x)\nabla u_i) &= 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega \\
u_i(0,\cdot) &= u^0 \in L^2(\Omega) \\
\partial_t u_i(0,\cdot) &= \tilde{u}^0 \in H^{-1}(\Omega), \quad i = 1, 2. 
\end{align*}
$$

In order to avoid technical problems related to boundary conditions and reflections on it, throughout the paper the space domain $\Omega$ is assumed to be a compact manifold without a boundary. Similarly, the coefficients entering the equation are taken regular enough ($C^{1,1}$ or better), thus ensuring well posedness of the problems under consideration, as well as of the bicharacteristic flow.

We investigate the conditions under which one gets an observability estimate for the averaged solution $u_1 + (1 - \theta)u_2$, with a parameter $\theta \in [0,1]$. The same conditions (and the same procedure) also imply an estimate for the difference of the solutions. More precisely the following theorem holds.

**Theorem 2.1** Suppose the equations’ coefficients satisfy

$$
\text{sign} \left( c_1(t,x) - c_2(t,x) \right) \neq 0, \quad (t,x) \in (0,T) \times \omega,
$$

where $\omega$ is an open subset of $\Omega$, and $T$ a positive constant such that $(0,T) \times \omega$ satisfies the GCC for the problem solved by $u_1$.

Then, for any $\theta_* > 0$ there exists a constant $C$ such that for all $\theta \in [\theta_*,1]$ the following estimate holds

$$
E(0) := \|u_1^0\|_{L^2}^2 + \|\tilde{u}^0\|_{H^{-1}}^2 \leq C \int_0^T \int_{\omega} |\theta u_1 \pm (1 - \theta)u_2|^2 dx dt.
$$

**Proof:**

Assume the contrary. Then there exists a sequence $(\theta^n)$ of numbers from $[\theta_*,1]$, as well as sequences of initial conditions $(u_1^{0n}, \tilde{u}_i^{0n})$, $i = 1, 2$ such that for corresponding solutions $u_1^n, u_2^n$ we have

$$
E^n(0) > n \int_0^T \int_{\omega} |\theta^n u_1^n \pm (1 - \theta^n)u_2^n|^2 dx dt.
$$

Without losing generality we can assume that $E^n(0) = 1$, and that (maybe after passing to a subsequence) $\theta^n$ converge to $\theta > 0$. 

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Thus (3) implies convergence (to zero) of the integral on the right hand side, and an H-measure $\nu$ associated to a (sub)sequence (of) $(\theta^n u_{1i}^0 \pm (1-\theta^n) u_{2i}^0)$ equals zero on $\langle 0, T \rangle \times \omega$. Furthermore, it is of the form

$$\nu = \theta^2 \mu_1 + (1-\theta)^2 \mu_2 \pm \theta(1-\theta)2\mathbb{R}\mu_{12}$$

where on the right hand side are listed the elements of the matrix measure associated to a vector (sub)sequence (of) $(u_{1i}^0, u_{2i}^0)$, with $\mu_{12}$ denoted the off-diagonal element.

According to the localisation property for H-measures, $\mu_1$ is supported within the set $\{\tau^2 + c_1(t, x)\xi_2 = 0\}$, while $\mu_2$ has its support inside $\{\tau^2 + c_2(t, x)\xi_2 = 0\}$. Due to the separation of velocities on the observability region, it follows that their support are disjoints, and the diagonal domination of H-measures implies $\mu_{12} = 0$.

Thus we get that

$$\nu = \theta^2 \mu_1 + (1-\theta)^2 \mu_2 = 0 \quad \text{on} \quad \langle 0, T \rangle \times \omega.$$  

As $\mu_1$ and $\mu_2$ are positive measures and $\theta > 0$, it follows that $\mu_1$ is zero on $\langle 0, T \rangle \times \omega$ as well. Using that $\langle 0, T \rangle \times \omega$ satisfies GCC for the problem (1) with $i = 1$, the propagation principle for H-measures implies that $\mu_1$ equals zero everywhere, which contradicts the assumption of initial conditions $u_{0n}^0, \bar{u}_{0n}^0$ having the constant, non-zero energy norm.

The observability inequality (2) of the last theorem is equivalent to the averaged control of a linear combination of solutions to the adjoint system. Proceeding similarly as in [9] one gets that the corresponding dual problem in the control theory consists of the following system of equations

$$\begin{align*}
\partial_t v_i - \text{div} \, \langle c_i(t, x) \nabla v_i \rangle &= \chi_{\{0, T\} \times \omega} f, \quad (t, x) \in \mathbb{R}^+ \times \Omega \\
v_i(0, \cdot) &= \bar{v}_{0i}^0 \in \mathcal{H}^1(\Omega) \\
\partial_t v_i(0, \cdot) &= \bar{v}_{0i}^1 \in \mathcal{L}^2(\Omega), \quad i = 1, 2 \tag{4}
\end{align*}$$

controlled by the same control $f \in \mathcal{L}^2(\mathbb{R}^+ \times \Omega)$. The last theorem implies that for any choice of initial data, and any final target couple $(v^T, \bar{v}^T) \in \mathcal{H}^1(\Omega) \times \mathcal{L}^2(\Omega)$ there exists a control $f$ such that

$$(\alpha v_1 + \beta v_2)(T, \cdot) = v^T, \quad \partial_t (\alpha v_1 + \beta v_2)(T, \cdot) = \bar{v}^T,$$

where $\alpha \neq 0$ and $\beta$ are arbitrary real constants.

**Remark 1** Several remarks are in order.
- The statement (and the proof) of the last theorem holds, under same conditions, if, instead of the same initial data for two problems, one assumes the data have just the same norm. However, for such a generalisation we do not find a counterpart problem in the control theory.
- The purpose of the lower bound $\theta_*$ in the theorem is to get a uniform observability constant. Otherwise, one can get the result for any choice of $\theta \in [0, 1]$, but with a constant depending on $\theta$.
- The result is easily generalised to any finite number of equations. More precisely, let us take a family of $N$ equations of type (1) with coefficients satisfying

$$\text{sign} \, (c_i(t, x) - c_i(t, x)) \neq 0, \quad (t, x) \in \langle 0, T \rangle \times \omega, \, i \neq 1,$$

where $\langle 0, T \rangle \times \omega$ satisfies the GCC for the first problem. Then for any averaging set of numbers $\theta_i \in [0, 1], i = 1..N$ such that $\sum_i \theta_i = 1$ and $\theta_1 > 0$ there exists a constant $C_{\theta_1}$ such that the following estimate holds

$$E(0) \leq C_{\theta_1} \int_0^T \int_\omega \left| \sum_{i=1}^N \theta_i \nabla v_i \right|^2 dx \, dt.$$
3. Simultaneous observability

We consider two problems for the wave equation with solutions denoted respectively by $u_1$ and $u_2$:

$$
\begin{align*}
\partial_{tt} u_i - \text{div} \left( c_i(t,x) \nabla u_i \right) &= 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\
\partial_t u_i(0,\cdot) &= u_i \in L^2(\Omega), \\
\partial_t u_i(0,\cdot) &= \tilde{u}_i \in H^{-1}(\Omega), \quad i = 1, 2.
\end{align*}
\tag{5}
$$

Unlike in the previous subsection, the initial data for two equations are arbitrary and not related in any manner.

Proceeding similarly as above, one obtains the following result.

**Theorem 3.1** Suppose the coefficients of (5) satisfy

$$
\text{sign} \left(c_1(t,x) - c_2(t,x)\right) \neq 0, \quad (t,x) \in (0,T) \times \omega,
$$

where $(0,T) \times \omega$ satisfies the GCC for the both problems in (5). Then, for any $\theta_* > 0$ there exists a constant $C$ such that for all $\theta \in [\theta_*, 1 - \theta_*]$ the following estimate holds

$$
E_1(0) + E_2(0) \leq C \int_0^T \int_\omega |\theta u_1 + (1 - \theta) u_2|^2 dx dt,
$$

where

$$
E_i(0) := \|u_i^0\|_{L^2}^2 + \|\tilde{u}_i\|_{H^{-1}}, \quad i = 1, 2
$$

stands for the initial energy of the system (5).

We refer to the last result as an simultaneous observability one, in accordance to the terminology in the dual theory. The corresponding dual problem consists of controlling each individual component of the system (4) by means of the same control.

The theorem also holds for an arbitrary finite number of equations assuming the GCC is satisfied for each problem considered, and all the velocities are strictly separated on the control region.

By using transmutation procedure from heat processes to waves presented in [5] the last result can be generalised to a system of heat equations:

$$
\begin{align*}
\partial_t z_i - \text{div} \left( c_i(x) \nabla z_i \right) &= 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\
z_i(0,\cdot) &= z_i^0 \in L^2(\Omega), \quad i = 1, 2.
\end{align*}
\tag{6}
$$

Applying the above theorem to transmutators

$$
u_i(s,x) = \int_{\mathbb{R}^+} k(t,s) z_i(t,x) dt,
$$

where the kernel $k$ is given by

$$
k(t,s) = \frac{1}{\sqrt{4\pi t}} \sin \left( \frac{ss}{4t} \right) e^{-\frac{s^2 + \omega^2}{4t}},
$$

and $S > 0$ is arbitrary, one gets the following result.

**Corollary 3.2** Assume that $\omega$ satisfies the GCC for the both problems for the wave equation solved by $u_i$, $i = 1, 2$. Then, for any $\theta_* > 0$ there exists a constant $C$ such that for all $\theta \in [\theta_*, 1 - \theta_*]$ the following estimate holds

$$
\sum_i \int_0^T \int_\omega e^{-\frac{4\pi^2}{t}} \|z_i(t,\cdot)\|_{L^2}^2 dx dt \leq C(T) \int_0^T \int_\omega \left| \left( \theta z_1 + (1 - \theta) z_2 \right)(t,x) \right|^2 dx dt.
\tag{7}
$$

The corollary enables one to estimate the energy of the system (6) at an arbitrary strictly positive time. However, the initial energy can not be recovered from (7). Furthermore, the constant $T$ can be arbitrary.
as well, which is in accordance with an infinite speed of propagation for the heat equation. Unlike it, the assumption on the set $\omega$ satisfying the GCC seems as an unnatural, technical requirement, and going beyond it is an open problem.

4. Conclusion

Results presented above can also be generalised to a case of infinitely many equations, with coefficients depending on a discrete, or even continuous parameter. Such generalisations are not straightforward and one has to pay special attention to the arguments based on localisation principle for H-measures. In particular, when averaging an infinite number of sequences, the measure associated to the average does not have to be supported within the set containing supports of H-measures associated to each particular sequence, which disables kind of arguments used in the proof of Theorem 2.1. Similarly, condition on separation of velocities requires more detailed analysis.

For a system of heat equations, better kind of estimates than those resulting from Corollary 3.2 can be obtained by using analysis elaborated in [6] for the case of constant coefficients. Determining and interpreting conditions under which the observability estimates can be related to averaged control of a finite system of heat equations remains an open problem.

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